Lecture 1 : Overview

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1.1 Minimal spanning trees and forests

Let $G = (V, E)$ be a finite graph and let $\{w_e\}_{e \in E}$ be distinct real weights on the edges. The *minimal spanning tree* (MST for short) of the graph is the spanning tree T, for which the sum of weights $w(T) = \sum_{e \in T} w(e)$ is minimal amongst all spanning trees of G. Assume from now on that $\{w_e\}_{e \in E}$ are i.i.d. uniform on $[0, 1]$.

Let G be a connected, infinite graph with finite degrees and fix two vertices v, w in it. Let G_n be a sequence of induced finite subgraphs such that $\cup_n G_n = G$ and let L_n be the distance in between v and w in the MST on G_n.

Question. Is $\{L_n\}$ a tight sequence? I.e., is true that as $M \to \infty$,

$$\sup_n \mathbb{P}(L_n > M) \to 0\,?$$

The following exercise gives an equivalent definition of a minimal spanning tree.

Exercise. Let E_H be the set of edges that are "heaviest in a cycle", i.e.

$$E_H = \{e : \exists \text{ cycle } e = e_1, e_2, \ldots, e_k \text{ with } w_e = \max_{i \le k} w_{e_i}\}\,.$$

Then $T = E - E_H$ is the minimal spanning tree.

Using the exercise we define the *free minimal spanning forest* (FMSF for short) of an infinite, connected graph G with finite degrees simply as the forest obtained by removing the edges E_H from the graph. It is easy to observe that this has no finite connected components. The event that the FMSF is connected is has probability 0 or 1 by translation invariance. It is also known that with probability 1, the FMSF of $\mathbb{Z}^2$ is connected.

Question. Is the FMSF of $\mathbb{Z}^d$ connected for $d \ge 3$?
Question. Is it true that for every transitive graph G we have that with probability 1 the number of components is either 1 or ∞?

1.2 Ising and Potts Model, especially on trees

Consider the problem of broadcasting on trees. Assume T is a rooted b-ary tree of depth n, assign the root a random value from $\{1\ \ldots, q\}$ for some fixed integer q. We assign each vertex in the tree a value in the following manner. Let $\epsilon > 0$, and for each child of the root copy the value of the root with probability $1-\epsilon$, or, with probability ϵ draw a uniform random value for it from $\{1\ \ldots, q\}$. Continue recursively to assign values to all vertices of the tree, including the leaves. We say *reconstruction* is possible if there is some algorithm that reads the values of the leaves and outputs a number from $\{1 \ldots, q\}$ such that the probability of this number being the root value is greater than $1/q + \delta$ for some $\delta > 0$ not depending on n.

It is known (see [1]) that there exists $\epsilon_c(b, q)$ such that for any $\epsilon > \epsilon_c(b, q)$ reconstruction is impossible, and for any $\epsilon < \epsilon_c(b, q,)$ reconstruction is possible. Furthermore,

$$1 - 2\epsilon_c(2, b) = \frac{1}{\sqrt{b}} .$$

Question. What is $\epsilon_c(q, b)$ for $q > 2$? It is conjectured that is satisfies

$$1 - q\epsilon_c(q, b) = \frac{1}{\sqrt{b}} .$$

The conjecture arises because of the following. Assume that the only information one receives is how many leaves are there of each kind. In this model, reconstruction also exhibits a phase transition (see [1]), denote this point by ϵ_{census}, which satisfies

$$1 - q\epsilon_{\text{census}} = \frac{1}{\sqrt{b}} .$$

References

[1] Mossel E. and Peres Y. (2003), Information flows on trees, *Ann. Appl. Probab.* **13** no. 3 817–844.