STAT 260B: Open problems in probability 3

Lecture 3

Lecture date: Jan 24 *Scribe: Ron Peled*

1 Free minimal spanning forest, union with percolation and Wired minimal spanning forest

Let $G = (V, E)$ be a connected, locally finite graph. Let us define the Free Minimal Spanning Forest (FMSF) on G.

The FMSF is a random subset of the edges of G, forming a forest. To define the FMSF first attach to each edge $e \in E$ a random label $U(e)$ such that $\{U(e)\}_{e\in E}$ are IID uniform $[0, 1]$. Then calculate for each $e \in E$:

$$Z(e) = \inf_{\varphi=(e_1,\ldots,e_k)} \max_{1\le i\le k} U(e_i)$$

Where the infimum runs over all finite paths in G connecting the end points of e and not containing e itself. If there are no such paths we let $Z(e) = 1$. Then the FMSF is $\{e \in E\ | U(e) < Z(e)\}$.

Some facts about the FMSF (exercises):

- The FMSF of a finite graph G is a tree, called the minimal spanning tree (MST).
- Suppose $G_1 = (V_1, E_1)$ is an induced subgraph of $G_2 = (V_2, E_2)$, that is, $V_1 \subseteq V_2$ and $E_1 = \{(u, v) \in E_2 \mid u, v \in V_1\}$.

  Consider the FMSF's F_1 and F_2 of G_1 and G_2 respectively and couple them together by using the same random labels $\{U(e)\}$ for both. We easily see that $F_2 \cap E_1 \subseteq F_1$, since if an edge $e \in F_2 \cap E_1$ it means that in G_2 there is no path connecting the end points of e all of whose labels are smaller than $U(e)$. In this case there is certainly no such path in the subgraph G_1.

  In particular, if $G = (V, E)$ is an infinite graph and $G_1 \subseteq G_2 \subseteq \ldots$ where $G_i = (V_i, E_i)$ is an infinite increasing sequence of induced finite subgraphs of G with $\cup_i V_i = V$ and with each G_i being connected. Then the MST's F_i in G_i (coupled by having all use the same labels $\{U(e)\}$) converge monotonically to the FMSF in G in the sense that for each edge $e \in E$. For all i such that $e \in E_i$, the indicator $1_{\{e\in F_i\}}$ is monotonically decreasing and $1_{\{e\in F_i\}} \to 1_{\{e\in F\}}$ (a.s.).

*<u>Exercise</u>: Find an infinite graph G where the FMSF is not connected (to be solved next lecture).

A non elementary example of a solution to the exercise is the graph $T \times \mathbb{Z}$ where T is a regular tree and the product graph has as vertex set the product of the two vertex sets and has an edge between (t, m) and (s, n) iff either $t = s$ and m is connected to n in $\mathbb{Z}$ or $m = n$ and t is connected to s in T.

Pemantle and Peres [1] showed that this graph has infinitely many components to its FMSF by showing that there can be no automorphism invariant (in distribution) random tree on this graph.

***<u>Exercise</u>: Find a direct proof of the fact that the FMSF has infinitely many components for $T \times \mathbb{Z}$.

We also mention a related "construction" which illustrates some of the pitfalls of proving existence of continuous time processes. Given a graph $G = (V, E)$ and random labels $\{U(e)\}$ on its edges as before. One may try to define a continuous time process indexed by the time set $T = [0, 1]$ which at any time t is a random subgraph of G, such that at time $t \in T$ we add all the edges $(u, v) \in E$ which have $U(u, v) = t$ and with the endpoints u and v not already connected by some path at a previous time. On the one hand, intuition says that this should give the FMSF of G at time $t = 1$, but on the other hand, the subgraph obtained at each time is clearly connected.

The paradox is resolved by noting that such a process does not actually exist. One needs to be careful when trying to construct a process which at each time can add infinitely many edges according to global dependencies on the subgraph constructed up to that time. Such a process may not exist.

We now prove a theorem concerning the union of the FMSF with independent bond percolation:

Theorem 1 *Let $G = (V, E)$ be a connected, locally finite graph. For $\epsilon > 0$, let $W_\epsilon \subseteq E$ be an independent bond percolation on G with probability ϵ of an open edge. That is, $W_\epsilon \subseteq E$ and each $e \in G$ satisfies $\mathbf{P}(e \in W_\epsilon) = \epsilon$ independently of other edges. Let F be the FMSF of G and suppose that W_ϵ and F are independent, then $W_\epsilon \cup F$ is a.s. connected.*

Proof: The key step is to observe that for $e \in E$, given $Z(e)$ and that $e \notin F$ then $U(e) \sim \text{uniform}(Z(e), 1)$. This is intuitively obvious and we shall not prove it here. We note also that this remains true conditioned on $\{U(e_j)\}$ for finitely many $e_j \notin F$.

It follows that we can couple the bond percolation and the FMSF of G in such a way that

$$\begin{aligned} F^{(\epsilon)} := F \cup W_\epsilon = F \cup \{e \in E \mid Z(e) \le U(e) < Z(e) + \epsilon(1 - Z(e))\} = \\ = \{e \in E \mid 1 - U(e) > (1 - \epsilon)(1 - Z(e))\} \end{aligned} \tag{1}$$

Where we've used the fact that a.s. $Z(e) \neq 1$ for all $e \in E$.

Now note that since G is connected, it is enough to show for any edge $(u,v) \in E$ that u and v are connected in $F^{(\epsilon)}$. Fix a particular realization of the labels $\{U(e)\}$. Take an edge $e = (u,v) \notin F^{(\epsilon)}$, then by (1) we know

$$1 - U(e) < (1-\epsilon)(1 - Z(e)) \tag{2}$$

Since a.s. equality does not hold. And by the definition of $Z(e)$, this means that we have a path $\varphi = (e_1, e_2, \ldots, e_k)$ connecting u and v for which $\max_i U(e_i)$ is so close to $Z(e)$ that

$$\frac{1-U(e)}{1-\epsilon} < \min_i (1 - U(e_i)) \tag{3}$$

Now if all $\{e_i\}$ are in $F^{(\epsilon)}$ we are done. Otherwise there again exists a j with $Z(e_j) < U(e_j)$, and then we have another path $\varphi = (e_1^2, e_2^2, \ldots, e_{k_2}^2)$ connecting the edges of e_j and satisfying$\frac{1-U(e_j)}{1-\epsilon} < \min_i (1 - U(e_i^2))$. Combining with (3) we get

$$\frac{1-U(e)}{(1-\epsilon)^2} < \min_i (1 - U(e_i^2))$$

Again, if all $\{e_i^2\}$ are in the $F^{(\epsilon)}$ we are done, otherwise we iterate the preceding procedure. After the k'th iteration we get

$$\frac{1-U(e)}{(1-\epsilon)^k} < \min_i (1 - U(e_i^k)) \tag{4}$$

But this inequality shows that we must stop at a finite stage k since $\frac{1}{1-\epsilon} > 1$ and so the left hand side will eventually be bigger than 1. This finishes the proof of the theorem. □

*Exercise: Show that this is false if W_ϵ is just an invariant ergodic percolation with marginals ϵ.

Definition 2 *For a connected graph $G = (V,E)$, the Wired Minimal Spanning Forest (WMSF) F_w in G consists of the edges $\{e \in E \mid U(e) < Z_w(e)\}$ with*

$$Z_w(e) = \min(Z(e), \inf \sup_j U(e_j))$$

Where $Z(e)$ is the same as in the FMSF and to describe the inf*, let P_u be the collection of all paths from u to infinity, then the* inf *is taken over $\{p_1 \cup p_2 \mid p_1 \in P_u, p_2 \in P_v, p_1 \cap p_2 = \emptyset, e \notin p_1 \cup p_2\}$. That is all paths between u and v which "go through infinity" and do not contain e.*

Note that unlike the FMSF, the WMSF in a tree does not have to be a tree. And furthermore if a percolation which removes edges with probability ϵ was made on the WMSF, only finite components would remain.

References

[1] Pemantle, R. and Peres, Y. *Nonamenable products are not treeable*. Israel J. Math. 118 (2000), 147–155.