Lectures 14,15

The game of battleship and salvo:

X	X	
	B	

A battleship is located on two adjacent squares of a three-by-three grid, shown by the two Xs in the example. A bomber, who cannot see the submerged craft, hovers overhead. He drops a bomb, denoted by B in the figure, on one of the nine squares. He wins if he hits and loses if he misses the submarine. There are nine pure strategies for the bomber, and twelve for the submarine. That means that the payoff matrix for the game is pretty big. We can use symmetry arguments to simplify the analysis of the game.

Indeed, suppose that we have two bijections

$$g_1 : \{ \text{moves of } I \} \to \{ \text{moves of } I \}$$

and

$$g_2 : \{ \text{moves of } II \} \to \{ \text{moves of } II \},$$

for which the payoffs a_{ij} satisfy

$$a_{g_1(i),g_1(j)} = a_{ij}. \tag{1}$$

If this is so, then there are optimal strategies for player I that give equal weight to $g_1(i)$ and i for each i. Similarly, there exists a mixed strategy for player II that is optimal and assigns the same weight to the moves $g_2(j)$ and j for each j.

In the example, we may take g_1 to the the map that flips the first and the third columns. Similarly, we take g_2 to do this, but for the battleship location. Another example of a pair of maps satisfying (1) for this game: g_1 rotates the bomber's location by 90 degrees anticlockwise, whereas g_2 does the same for the location of the battleship. Using these two symmetries, we may now write down a much more manageable payoff matrix:

SHIP BOMBER	centre	off-centre
corner	0	1/4
midside	1/4	1/4
middle	1	0

These are the payoff in the various cases for play of each of the agents. Note that the pure strategy of corner for the bomber is this reduced game in fact corresponds to the mixed strategy of bombing each corner with 1/4 probability in the original game. Similarly, for each of the pure strategies in the reduced game.

We use domination further to simplify things. For the bomber, the strategy 'midside' dominates that of 'corner'. We have busted down to

SHIP BOMBER	centre	off-centre
midside	1/4	1/4
middle	1	0

Now note that for ship - that is trying to escape the bomb and thus is heading away from the high numbers on the table - off-centre dominates centre

SHIP BOMBER	off-centre
midside	1/4
middle	0

The bomber picks the better alternative - technically, another application of domination - and picks midside over middle. The value of the game is 1/4, the bomb drops on one of the four middles of the sides with probability 1/4 for each, and the submarine hides in one of the eight possible locations that exclude the centre, choosing any given one with a probability of 1/8. *Preparing for the proof of the minimax theorem:* We mentioned that convex

geometry plays an important role in the Von Neumann minimax theorem. Recall that:

Definition 1 *A set $K \subseteq \mathbb{R}^d$ is convex if, for any two points $\mathbf{a}, \mathbf{b} \in K$, the line segment that connects them,*

$$\{p\mathbf{a} + (1-p)\mathbf{b} : p \in [0,1]\}$$

also lies in K.

The main fact about convex sets that we will need is:

Theorem 1 (Separation theorem for convex sets) *Suppose that $K \subseteq \mathbb{R}^d$ is closed and convex. If $\mathbf{0} \notin K$, then there exists $\mathbf{z} \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that*

$$0 < c < \mathbf{z}^T\mathbf{v}\,,$$

for all $\mathbf{v} \in K$.

What the theorem is saying is that there is a hyperplane that separates $\mathbf{0}$ from K: this means a line in the plane, or a plane in $\mathbb{R}^3$. The hyperplane is given by

$$\{\mathbf{x} \in \mathbb{R}^d : z^T.x = c\}.$$

The theorem implies that on any continuous path from $\mathbf{0}$ to K, there is some point that gets mapped into the hyperplane.

Proof: We may find $\mathbf{z} \in K$ for which

$$||\mathbf{z}|| = \sqrt{\sum_{i=1}^{d} z_i^2} = \inf_{\mathbf{v} \in K} ||\mathbf{v}||.$$

This is because the function $K \cap \{\mathbf{x} \in \mathbb{R}^d : ||x|| \leq R\} \to [0, \infty) : \mathbf{v} \to ||\mathbf{v}||$ (with R large) is continuous, with its domain being a closed and bounded set. Therefore, it attains its infimum, at a point that we have called $\mathbf{z}$. Since $||\mathbf{z}|| \leq R$, there can be no point in the part of K not in the domain of this map with a lower norm. Choose $c = (1/2)||\mathbf{z}||^2 > 0$. We have to check that $c < \mathbf{z}^T.\mathbf{v}$ for each $\mathbf{v} \in K$. To do so, consider such a $\mathbf{v}$. For $\epsilon \in (0,1)$, we have that $\epsilon\mathbf{v} + (1-\epsilon)\mathbf{v} \in K$. Hence,

$$||\mathbf{z}||^2 \leq ||\epsilon\mathbf{v} + (1 - \epsilon\mathbf{v}||^2 = (\epsilon\mathbf{v}^T + (1-\epsilon)\mathbf{v}^T).(\epsilon\mathbf{v} + (1-\epsilon)\mathbf{v}^T),$$

the first inequality following from the fact that $\mathbf{z}$ has the minium norm of any point in K. We obtain

$$\mathbf{z}^T\mathbf{z} \leq \epsilon^2\mathbf{v}^t\mathbf{v} + (1-\epsilon)^2\mathbf{w}^T.\mathbf{w} + 2\epsilon(1-\epsilon)\mathbf{v}^T\mathbf{w}.$$

Multiplying out and cancelling an ϵ:

$$\epsilon(2\mathbf{v}^t\mathbf{z} - \mathbf{v}^t\mathbf{v} - \mathbf{z}^t\mathbf{z}) \leq 2(\mathbf{v}^t\mathbf{z} - \mathbf{z}^t\mathbf{z}).$$

Taking $\epsilon \downarrow 0$, we find that

$$0 \leq \mathbf{v}^t\mathbf{z} - \mathbf{z}^t\mathbf{z},$$

which implies that

$$\mathbf{v}^t\mathbf{z} \geq 2c > c,$$

as required. □