Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 18,19

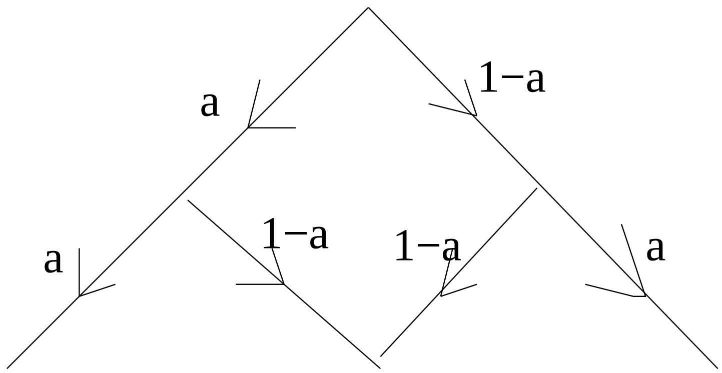
A discussion of the bomber and submarine game: Recall the bomber and submarine game: a submarine is located at a site in $\mathbb{Z}$ at any given time step in $\{0, 1, \ldots\}$, and moves either left or right for the next time step. In the game G_n, the bomber drops one bomb at some time $j \in \{0, 1, \ldots, n\}$. The bomb arrives at time $j + 2$, and destroys the submarine if it hits the right site. What is the value of the game G_n? The answer depends on n. The value of G_0 is one, because the submarine may ensure that it has a 1/3 probability of being at any of the sites -2,0 or 2 at time 2. It moves left or right with equal probability at the first time step, and then turns with probability of 1/3. The value of 1/3 for the game G_1 can be obtained by pursuing this strategy. We have already decided how the submarine should move in the first two time steps. If the submarine is at 1 at time 1, then it moves to 2 at time 2 with probability 2/3. Thus, it should move with probability 1/2 to each of sites 1 or 3 at time 2 if it is at site 2 at that time, to ensure a value of 1/3 for G_1. This forces it to move from site 0 to site 1 with probability 1, if it visited site 1 at time 1. Obtaining the same values in the symmetric case where the submarine moves through site -1 at time 1, we obtain a strategy for the submarine that ensures that it is hit with a maximum probability of 1/3 in G_1.

It is impossible to pursue this strategy to obtain a value of 1/3 for the game G_2. Indeed, $v(G_2) > 1/3$. We now describe Isaac's strategy for the game. It is not optimal in any given game G_n, but it does have the merit of having the same limiting value, as $n \to \infty$, as optimal play. In G_0, the strategy is as shown:

The rule in general is: turn with a probability of $1 - a$, and keep going with a probability of a. The strategy is simple in the sense that the transition rates in any 2-play subtree of the form in the figure has the transition rates shown there, or its mirror image. We now choose a to optimize the probability of evasion for the submarine. Its probabilities of arrival at sites -2,0 or 2 at time 2 are a^2, $2a(1 - a)$ and $a(1 - a)$. We have to choose a so that $\max\{a^2, 1 - a\}$ is minimal. This value is achieved when $a^2 = 1 - a$, whose solution in $(0, 1)$ is given by $a = 2/(1 + \sqrt{5})$.

The payoff for the bomber against this strategy is at most $1 - a$. We have proved that the value $v(G_n)$ of the game G_n is at most $1 - a$, for each n.

Konig's lemma: We now prove a result by using Hall's marriage lemma.



Lemma 1 *Consider an $n \times m$ matrix whose entries consist of zeros and ones. Call two ones independent if no row nor column contain them both. A cover of the matrix is a collection of rows and column whose union contains each of the ones. Then: the maximal size of a set of independent ones is equal to the minimal size of a cover.*

Proof: That the maximal size of a set of independent ones is at most the minimal size of a cover is easy: each one in the independent set is covered by a line, and no two are covered by the same line. Consider a maximal independent set of ones (of size k), and a minimal cover consisting of l lines. Suppose that among these l lines, there are r rows and c columns. In applying Hall's lemma, the rows correspond to the boys and columns not in the cover to girls. A row knows such a column if their intersection contains a one from the independent set.

Suppose that j of these rows know $s < j$ columns not in the minimal cover. We could replace these j rows by these s columns to obtain a smaller cover. This is impossible, meaning that we know that every set of j rows knows at least j columns not in the minimal cover. By Hall's lemma, we can match up the r rows with columns outside the cover and known to them.

Similarly, we obtain a $1-1$ matching of the c columns in the cover with c rows outside the cover. Each of the intersections of these c matched rows and columns contains a 1. Similarly, with the r matched rows and columns just constructed. The $r+c$ resulting ones are independent. This completes the proof. □

We use Konig's lemma to analyse a *hide and seek* game in a matrix. In this game, a matrix whose entries are zeros and ones is given. Player I chooses a one somewhere in the matrix, and hides there. Player II chooses a row or column and wins a payoff of 1 if the line that he picks contains the location chosen by player I. A strategy for player I is to pick a maximal independent set of ones, and then hide in a uniformly chosen element of it. A strategy for player II consists of picking uniformly at random one of the lines of a minimal cover of the matrix. Konig's lemma shows that this is a joint optimal strategy, and that the value of the game is k^{-1}, where k is the size of the maximal set of independent ones.