Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 23,24,25

General sum games: We now turn to discuss the theory of general sum games, in which some degree of cooperation between the players may be optimal. A general sum game is given in strategic form by two matrices A and B, whose entries give the payoffs of given joint pure strategies to each of the two players. Here are two examples:

Example: the prisoner's dilemma: Two suspects are held and questioned by police who ask each of them to confess or to remain silent. If both confess, they will each spend a year in prison. If both remain silent, five years to each is the sentence. If one confesses and the other is silent, then the first is sentenced to ten years, and the other goes free. Writing the payoff as the number of years that remain in the following decade apart from those spent in prison, we obtain the following payoff matrix:

II / I	S	C
S	(5,5)	(0,10)
C	(10,0)	(1,1)

The pay-off matrices for players I and II are the 2×2 matrices given by the collection of first, or second, entries in each of the vectors in the above matrix.

If the players only play one round, then there is an argument involving domination that each should confess, in the sense that the outcome she secures is preferable to the alternative of remaining silent, whatever the behaviour of the other player. However, this outcome is much worse than that secured for each player by both remaining silent. In a once-only game, the 'globally' preferable outcome of each remaining silent could only occur were each player to suppress the desire to achieve the best outcome in selfish terms. In games with repeated play ending at a known time, the same applies, by an argument of backward induction. In games with repeated play ending at a random time, however, the globally preferable solution may arise even with selfish play.

Example: the battle of the sexes: The wife wants to head to the opera but the husband yearns instead to spend an evening watching baseball. Neither is satisfied by an evening without the other. In numbers, here's the scenario:

II / I	O	B
O	(4,1)	(0,0)
B	(0,0)	(1,4)

(so that player I is the wife and II, the husband.)

Player I can guarantee a safety value of $\max_{p\in\Delta_n}\min_{q\in\Delta_n} p^T Bq$, where B denotes the matrix of payoffs received by player I. However, a minimax approach is unsuitable in this context. We now introduce a central notion for the study of general sum games:

Definition 1 (Nash equilbrium) *A pair of vecotrs (x^*, y^*) with $x^* \in \Delta_m$ and $y^* \in \Delta_n$ define a Nash equilibrium, if no players gains by deviating unilaterally from it. That is,*

$$x^{*T}Ay^* \geq x^T Ay^*$$

for all $x \in \Delta_m$, and

$$x^{*T}By^* \geq x^T By^*$$

for all $y \in \Delta_n$.

There can be many Nash equilibria in a game. If x and y are unit vectors, with a 1 in some coordiante and 0 in the others, then the equilbrium is called *pure*. In battle of the sexes, there are two pure equilibria: these are BB and OO. There is also a mixed equilibrium, $(4/5, 1/5)$ and $(1/5, 4/5)$, having the low value of $4/5$. In this example, it is quite artificial to suppose that the two players cannot discuss, and that there are not repeated plays.
Example: a case of hide-and-seek: Consider the hide-and-seek game with payoff matrix B given by

$$\begin{vmatrix} 1 & 1/2 \\ 1/3 & 1/5 \end{vmatrix}$$

This means that the matrix D is equal to

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

To determine a minimal cover of the matrix D, consider first a cover that has all of its mass on the rows: $u = (2, 5)$ and $v = (0, 0)$. Note that rows 1 and 2 know only column 2, according to the definition of 'knowing' introduced in the analysis of this game. Modifying the vectors u and v according to the rule given in this analysis, we obtain updated vectors, $u = (1, 4)$ and $v = (0, 1)$, whose sum is 6, equal to the expression $\max_\pi D_\pi$ (obtained by choosing the permutation $\pi = id$.)

An optimal strategy for the hider is to play $p(1,1) = 1/6$ and $p(2,2) = 5/6$. An optimal strategy for the seeker consists of playing $q(row1) = 1/6$, $q(row2) = 2/3$ and $q(col2) = 1/6$. The value of the game is $1/6$.
Another example: lions and antelopes:

Antelopes have been observed to jump energetically when a lion nearby seems liable to hunt them. Why do they expend energy in this way? One theory was that the antelope were signalling danger to others at some distance, in a community-spirited gesture. However, the antelope have been observed doing this all alone. The currently accepted theory is that the signal is intended for the lion, to indicate that the antelope is in good health and is unlikely to be caught in a chase.

We now consider a simple model, where two cheetahs are giving chase to two antelope. The cheetahs will catch any antelope they choose. If they choose the same one, they must share the spoils. If they catch one without the other cheetah doing likewise, the catch is unshared. There is a large antelope and a small one, that are worth l and s to the cheetahs. Here is the matrix of payoffs:

II / I	L	S
L	(l/2,l/2)	(l,s)
S	(s,l)	(s/2,s/2)

If $l \geq 2s$, the first row dominates the second, and likewise, the columns. Each cheetah chases the larger antelope. If $s < l < 2s$, and the first cheetach chases the large antelope with probability x, then the expected payoff to the second cheetah of chasing the larger antelope is equal to

$$\frac{l}{2}x + (1-x)l,$$

and that arising from chasing the smaller antelope is

$$xs + (1-x)\frac{s}{2}.$$

A symmetric mixed Nash equilibrium arises at that value of x for which these last two quantities are equal, because, at any other value of x, player II would have cause to deviate from the mixed strategy $(x, 1-x)$ to the better of the pure strategies available. We find that

$$x = \frac{2l - s}{l + s}.$$

Symmetric mixed Nash equilibria are of particular interest. It has been experimentally verified that in some biological situations, systems approach such an equilibria, presumably by mechanisms of natural selection.

General sum games with $k \geq 2$ players: It doesn't make sense to talk about zero-sum games when there are more than two players. The notion of a Nash equilibrium, however, can be used in this context. We now describe formally the set-up of a game with $k \geq 2$ players. Each player i has a set S_i of pure strategies. If, for each $i \in \{1, \ldots, k\}$, player i uses strategy $l_i \in S_i$, then player j has a payoff of $F_j(l_1, \ldots, l_k)$, where we are given functions $F_j : S_1 \times S_2 \times \ldots \times S_k \to \mathbb{R}$, for $j \in \{1, \ldots, k\}$.

Example: an ecology game. Three firms will either pollute a lake in the following year, or purify it. They pay 1 unit to purify, but it is free to pollute. If two or more pollute, then the water in the lake is useless, and each firm must pay 3 units to obtain the water that they need from elsewhere. If at most one firm pollutes, than the water is usable, and the firms incur no further costs. Assuming that firm III purifies, the cost matrix is

II / I	Po	Pu
Pu	(1,0,1)	(1,1,1)
Po	(3,3,3+1)	(0,1,1)

If firm III pollutes, then it is

II / I	Po	Pu
Pu	(3+1,3,3)	(1,1,0)
Po	(3,3,3)	(3,3+1,3)

To discuss the game, we firstly introduce the notion of Nash equilibrium in the context of games with several players:

Definition 2 *A pure Nash equilibrium in a k-person game is a set of pure strategies for each of the players,*

$$(l_1^*, \ldots, l_k^*) \in S_1 \times \ldots \times S_k$$

such that, for each $j \in \{1, \ldots, k\}$ and l_j,

$$F_j(l_1^*, \ldots, l_{j-1}^*, l_j, l_{j+1}^*, \ldots, l_k^*) \leq F_j(l_1^*, \ldots, l_{j-1}^*, l_j^*, l_{j+1}^*, \ldots, l_k^*).$$

More generally, a mixed Nash equilibrium is a collection of k probability vectors $\tilde{X}^i$, each of length $|S_i|$, such that

$$F_j(\tilde{X}^1, \ldots, \tilde{X}^{j-1}, X, \tilde{X}^{j+1}, \ldots, \tilde{X}^k) \leq F_j(\tilde{X}^1, \ldots, \tilde{X}^{j-1}, \tilde{X}^j, \tilde{X}^{j+1}, \ldots, \tilde{X}^k),$$

for each probability vector X of length $|S_j|$. We have written:

$$F_j(X^1, X^2, \ldots, X^k) := \sum_{l_1 \in S_1, \ldots, l_k \in S_k} X^1(l_1) \ldots X^k(l_k) F_j(l_1, \ldots, l_k).$$

Definition 3 *A game is symmetric if, for every $i_0, j_0 \in \{1, \ldots, d\}$, there is a permutation π of the set $\{1, \ldots, d\}$ such that $\pi(i_0) = j_0$ and*

$$F_{\pi(i)}(l_{\pi(1)}, \ldots, l_{\pi(k)}) = F_i(l_1, \ldots, l_k).$$

For this definition to make sense, we are in fact requiring that the strategy sets of the players coincide.

We will prove the following result:

Theorem 1 *Nash's theorem Every game has a Nash equilibrium.*

Note that the equilbrium may be mixed.

Corollary 1 *In a symmetric game, there is a symmetric Nash equilibrium.*

Returning to the ecology game, note that the pure equilibria consist of each firm polluting, or one of the three firms polluting, and the remaining two purifying. We now seek a symmetric mixed equilibrium. There is just one parameter, $p \in (0, 1)$, which is the probability that a given firm will purify the lake. For firm 1, the expected cost from purifying is equal to $1+3(1-p)^2$, and from polluting is $3(1 - p^2)$. Equating these two expressions, we obtain that p is either one of the two values $1/2 + \sqrt{3}/6$, or $1/2 - \sqrt{3}/6$, each of which lies in the interval $(0, 1)$. These values correspond to the symmetric mixed equilibria for the game.