Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 26,27,28

Example: the game of chicken. We solve the game of chicken. The pure equilibria are (C, I) and (I, C). To determine the symmetric mixed equilibria, suppose that player I plays C with probability x and I with probability $1-x$. This presents player II with expected payoffs of $2x-1$ if she plays C, and $(a+2)x-a$ if she plays I. We seek an equilibrium where player II has positive weight on each of C and I, and thus one for which

$$2x - 1 = (a+2)x - a.$$

That is, $x = 1 - 1/a$. The payoff for player to is $2x-1$, which equals $1-2/a$.

There is an abstract paradox here. We have a symmetric game with payoff matrices A and B that has a unique symmetric equilibrium with payoff γ. By replacing A and B by smaller matrices $\tilde{A}$ and $\tilde{B}$, we obtain a payoff $\tilde{\gamma}$ in a unique symmetric equilibrium that exceeds γ.

The proof of Nash's theorem. Recall Nash's theorem:

Theorem 1 *For any general sum game with two players, there exists at least one Nash equilibrium.*

To prove this theorem, we will use:

Theorem 2 (Brouwer's fixed point theorem) *If $K \subseteq \mathbb{R}^d$ is closed, convex and bounded, and $T : K \to K$ is continuous, then there exists $x \in K$ such that $T(x) = x$.*

Remark. The fixed point theorem is easy in dimension $d = 1$, when K is a closed interval $[a, b]$. Defining $f(x) = T(x) - x$, note that $T(a) \geq a$ implies that $f(a) \geq 0$, while $T(b) \leq b$ implies that $f(b) \leq 0$. The intermediate value theorem assures the existence of $x \in [a, b]$ for which $f(x) = 0$, and thus, so that $T(x) = x$. Note also that each of the hypotheses on T in the theorem are required. Consider $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x+1$, as well as $T : (0, 1) \to (0, 1)$ given by $T(x) = x/2$, and also, $T : \{z \in \mathbb{C} : |z| \in [1, 2]\} \to \{z \in \mathbb{C} : |z| \in [1, 2]\}$ given by $T(x) = x \exp i\pi/2$.

Proof of Nash's theorem using Brouwer's theorem. Suppose that the game is specified by payoff matrices $A_{m \times n}$ and $B_{m \times n}$ for players I and II. We will define a map $T : K \to K$ (with $K = \Delta_m \times \Delta_n$) from a pair of strategies for the two players to another such pair. Note firstly that K is convex, closed and bounded. Define, for $x \in \Delta_m$ and $y \in \Delta_n$,

$$c_i(x, y) = c_i = \max\{ \sum_j a_{ij} y_j \, - \, x^T A y \, , \, 0\}.$$

Or:

$$c_i = \max\{A_i y \,-\, x^T A y\,,\,0\},$$

where A_i denotes the i-th row of the matrix A. That is, c_i is equal to the gain for player I obtained by switching from strategy x to pure strategy i, if theis gain is positive: otherwise, it is zero. Similarly, we define

$$d_j(x,y) = d_j = \max\{x^T B^{(j)} \,-\, x^T B y\,,\,0\},$$

where $B^{(j)}$ denotes the j-th column of B. The quantities d_j have the same interpretation for player II as the c_i do for player I. We now define the map T: it is given by $T(x,y) = (\tilde{x},\tilde{y})$, where

$$\tilde{x}_i = \frac{x_i + c_i}{1 + \sum_{k=1}^m c_k}$$

for $i \in \{1,\ldots,m\}$, and

$$\tilde{y}_j = \frac{y_j + d_j}{1 + \sum_{k=1}^n d_k}$$

for $j \in \{1,\ldots,n\}$. Note that T is continuous, because c_i and b_j are. Applying Brouwer's theorem, we find that ther exists $(x,y) \in K$ for which $(x,y) = (\tilde{x},\tilde{y})$. We now claim that, for this choice of x and y, each $c_i = 0$ for $i \in \{1,\ldots,m\}$, and $d_j = 0$ for $j \in \{1,\ldots,n\}$. To see this, suppose, for example, that $c_1 > 0$. Note that the current payoff of player I is a weighted average $\sum_{i=1}^m x_i A_i y$. There must exists $i \in \{1,\ldots,m\}$ for which $x^T A y \geq A_i y$, and $x_i > 0$. For this i, we have that $c_i = 0$. This implies that

$$\tilde{x}_i = \frac{x_i + c_i}{1 + \sum_{k=1}^m c_k} < x_i,$$

because $c_1 > 0$. That is, the assumption that $c_1 > 0$ has brought a contradiction.

We may repeat this argument for each $i \in \{1,\ldots,m\}$, thereby proving that each $c_i = 0$. Similarly, each $d_j = 0$. We deduce that $x^T A y \geq A_i y$ for all $i \in \{1,\ldots,m\}$. This implies that

$$x^T A y \geq {x^*}^T A y$$

for all $x^* in \Delta_m$. Similarly,

$$x^T A y \geq x^T A y^*$$

for all $y^* \in \Delta_n$. Thus, (x,y) is a Nash equilibrium. □

Example: the fish-selling game. Fish being sold at the market is fresh with probability 2/3 and old otherwise, and the customer knows this. The seller knows whether the particular fish on sale now is fresh or old. The customer asks the fish-seller whether the fish is fresh, the seller answers,

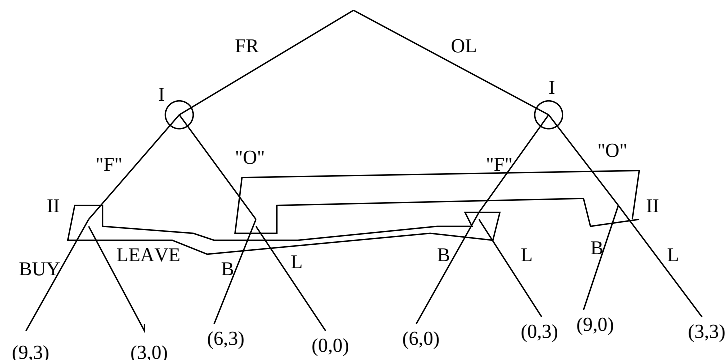

Figure 1: The Kuhn tree for the fish selling game.

and then the customer decides to buy the fish, or to leave without buying it. The payoff to the seller is 6 is the customer buys the fish, 0 if he leaves without buying it. Being truthful has a value of 3 to the seller. The customer has a payoff of 3 from buying the fish if it is fresh, or from leaving if it is old. The Kuhn tree for the game is depicted in the figure. Here are the payoffs for the two players.

II / I	BB	BL	LB	LL
FF	(8,2*)	(8*, 2*)	(2,1)	(2,1)
FO	(9*, 2)	(7,3*)	(5,0)	(3*, 1)
OF	(6,2)	(2,0)	(4,3*)	(0,1)
OO	(7,2*)	(1,1)	(7,2*)	(1,1)

We reduce the matrix by domination arguments. Note that telling the truth dominate lying as a strategy for the seller. In the reduced matricx, LL is dominated for the customer, as is LB. We have the matrix:

II / I	BB	BL
FF	(8,2)	(8,2)
FO	(9,2)	(7,3)
OO	(7,)	(1,1)

In this matrix, OO is dominated for the seller, at which point, BB is dominated for the customer. We obtain that a pure strategy of FF for the buyer and BL for the seller is optimal. Note that, in this case, the incenitve for truth-telling has not been strong enough to deflect the seller from his aim of selling fish: he will always claim that the fish is fresh.

Some fixed point theorems. We will discuss some fixed point theorems, beginning with:

Theorem 3 (Banach's fixed point theorem) *Let K be a complete metric space. Suppose that $T : K \to K$ satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in K$, with $0 < \lambda < 1$ fixed. Then T has a unique fixed point in K.*

Note: recall that a metric space is complete if each Cauchy sequence therein converges to a point in the space. Consider the metric space being a subset of $\mathbb{R}^d$ and the metric d being Euclidean distance if you are not familiar with these definitions.
Proof: Uniqueness of the fixed point: if $Tx = x$ and $Ty = y$, then

$$d(x, y) = d(Tx, Ty) \leq \lambda d(x, y).$$

Thus, $d(x, y) = 0$, and $x = y$. As for existence, given any $x \in K$, we define $x_n = Tx_{n-1}$ for each $n \geq 1$, setting $x_0 = x$. Set $a = d(x_0, x_1)$, and note that $d(x_n, x_{n+1}) \leq \lambda^n a$. If $k > n$, then

$$d(x_k, x_n) \leq d(x_n, x_{n+1}) + \ldots + d(x_{k-1}, x_k) \leq a(\lambda^n + \ldots + \lambda^{k-1}) \leq \frac{a\lambda^n}{1-\lambda}.$$

This implies that $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence. The metric space K is complete, whence $x_n \to z$ as $n \to \infty$. Note that

$$d(z, Tz) \leq d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tz) \leq (1 + \lambda)d(z, x_n) + \lambda^n a \to 0$$

as $n \to \infty$. Hence, $d(Tz, z) = 0$, and $Tz = z$. $\square$