**Stat 155 Game theory, Yuval Peres** **Fall 2004**

# Lectures 29,30,31

**Example regarding Banach's fixed point theorem.** Consider the map $T : \mathbb{R} \to mathbbR$ given by

$$T(x) = x + \frac{1}{1 + \exp(x)}.$$

Note that, if $x < y$, then

$$T(x) - x = \frac{1}{1 + \exp(x)} > \frac{1}{1 + \exp(y)} = T(y) - y,$$

implying that $T(y) - T(x) < y - x$. Note also that

$$T'(x) = 1 - \frac{\exp(-x)}{\left(1 + \exp(x)\right)^2} > 0,$$

so that $T(y) - T(x) > 0$. Thus, $T$ decreases distances, but it has no fixed point. This is not a counterexample to Banach's fixed point theorem, however, because there does not exist any $\lambda \in (0, 1)$ for which $|T(x) - T(y)| < \lambda|x - y|$ for all $x, y \in \mathbb{R}$.

**Theorem 1 (Compact fixed point theorem)** *If $X$ is a compact metric space and $T : X \to X$ satisfies $|T(x) - T(y)| < \lambda|x - y|$ for all $x, y \in X$, $x \neq y$, then $T$ has a fixed point.*

**Proof.** Let $f : X \to X$ be given by $f(x) = d(Tx, x)$. Note that $f$ is continuous:

$$|f(x) - f(y)| \leq |d(x, Tx) - d(y, Ty)| \leq d(x, y) + d(Tx, Ty) \leq 2d(x, y),$$

where the first inequality followed from two applications of the triangle inequality. By the compactness of $X$, there exists $x_0 \in X$ such that

$$f(x_0) = \inf_{x \in X} f(x). \tag{1}$$

If $Tx_0 \neq x_0$, then $d(x, Tx_0) > d(Tx_0, T^2x_0)$, contradiction (1). Thus implies that $Tx_0 = x_0$. $\square$

**Proof of Brouwer's theorem by means of Sperner's lemma.** We now state and prove a tool to be used in the proof of Brouwer's fixed point theorem.

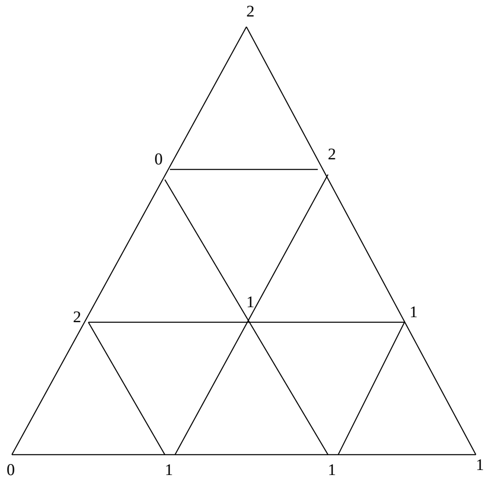

Figure 1: Sperner's lemma when $d = 2$.

**Lemma 1 (Sperner)** *In* $d = 1$*: suppose that the unit interval is subdivided* $0 = t_0 < t_1 < \ldots < t_n = 1$*, with each* $t_i$ *being marked zero or one,* $t_0$ *marked zero and* $t_1$ *marked one. Then the number of adjacent pairs* $(t_j, t_{j+1})$ *with different markings is odd.*

*In* $d = 2$*: subdivide a triangle into smaller triangles in such a way that a vertex of one small triangle may not lie in the interior of the edge of another. Label the vertices of the small triangles* 0*,* 1 *or* 2*. The three vertices of the big triangle must be labelled* 0,1 *and* 2*. Vertices of the small triangles that lie on the edge of the big triangle must receive a label that is equal to one or other of the labels given to the vertices of the big triangle that form the endpoints of the big side on which the vertex in question lies.*

**Remark.** Sperner's lemma holds in any dimension. In the general case $d$, we replace the triangle by a $d$-simplex, use $d$ labels, with analogous restrictions on the labels used.

**Proof.** For $d = 1$, this is obvious. For $d = 2$, we will count in two ways the set $Q$ of pairs of (small) triangle and edge on that triangle. Let $A_{12}$ denote the number of 12 type edges in the boundary of the big triangle. Let $B_{12}$ be the number of such edges in the interior. Let $N_{abc}$ denote the number of triangles where the three labels are $a$, $b$ and $c$. Note that

$$N_{012} + 2N_{112} + 2N_{122} = A_{12} + 2B_{12},$$

because each side of this equation is equal to the number of pairs of triangle and edge, where the edge is of type (12). From the case $d = 1$ of the lemma, we know that $A_{12}$ is odd (and in general, we may induct on the dimension, and use the inductive hypothesis to find that this quantity is odd.) $\square$

**Corollary 1 (No retraction theorem)** *Let $K \subseteq \mathbb{R}^d$ be compact and convex, and with non-empty interior. There exists no continuous map $f : K \to \partial K$ whose restriction to $\partial K$ is the identity.*

**Proof for $d = 2$.** Firstly, we show that it suffices to take $K = \Delta$, where $\Delta$ is an equilateral triangle. Form we may locate $x\ inK$ such that there exists a small triangle centred at $x$ and contained in $K$, because $K$ has a non-empty interior. We call this triangle $\Delta$ for convenience. Construct a map $h : K \to \Delta$ as follows. For each $y \in \partial K$, define $h(y)$ to be equal to that element of $\partial\Delta$ that the line segment from $x$ through $y$ intersects. Setting $h(x) = x$, define $h(z)$ for other $z \in K$ by a linear interpolation of the values $h(x)$ and $h(q)$, where $q$ is the element of $\partial K$ lying on the line segment from $x$ through $z$.

Note that, if $F : K \to \partial K$ is a retraction, then $h \circ F \circ h^{-1} : \Delta \to \delta\Delta$ is a retraction of $\Delta$. This is the reduction we claimed.

Now suppose that $F_\Delta : \Delta \to \partial\Delta$ is a retraction. SInce $F = F_\Delta$ is continuous and $\Delta$ is compact, there exists $\delta > 0$ such that $x, y \in \Delta$ satisfying $|x - y| < \delta$ also satisfy $|f(x) - f(y)| < \frac{\sqrt{3}}{4}$.

Triangulate $\Delta$ into triangles of sidelength less than $\delta$. In this subdivision, label any vertex $x$ according to the label of the label of the vertex of $\Delta$ nearest to $F(x)$, with an arbitrary choice being made to break ties (the vertices of $\Delta$ are labelled $0, 1, 2$.)

By Sperner's lemma, there exists a small triangle whose vertices are labelled $0, 1, 2$. The condition that $|f(x) - f(y)| < \frac{\sqrt{3}}{4}$ implies that any pair of these vertices must be mapped under $F$ to interior points of one of the side of $\Delta$, with a different side of $\Delta$ for each pair. This is impossible, implying that no retraction of $\Delta$ exists. $\square$

**Example to recall zero-sum games.** Consider the zero sum game whose payoff matrix is given by:

| II<br>I | | | |
|---|---|---|---|
| | 1 | 0 | 8 |
| | 2 | 3 | -1 |

To solve this game, firstly, we search for saddle points - a value in the matrix and is maximal in its column and minimal in its row. None exist in this case. Nor are there any evident dominations of rows or columns.

Suppose then that player $I$ plays the mixed strategy $(p, 1-p)$. If there is an optimal strategy for player $II$ in which she plays each of her three pure strategies with positive probability, then

$$2 - p = 3 - 3p = 9p - 1.$$

No solution exists, so we consider in turn mixed strategies for player $II$ in which one pure strategy is never played. If the third column has no weight, then $2-p=3-3p$ implies that $p=1/2$. however, the entry 3 in the matrix becomes a saddle point in the $2\times 2$ matrix formed by eliminating the third column, which is not consistent with $p=1/2$.

Consider instead strategies supported on columns 1 and 3. The equality $2-p=9p-1$ yields $p=3/10$, giving a payoff for player $II$ of

$$\Big(17/10, 27/10, 17/10\Big).$$

If player $II$ plays column 1 with probability $x$ and column 3 in the other case, then player $I$ sees the payoff vector $(8-7x, 3x-1)$. These quantities are equal when $x=9/10$, so that player $I$ sees the payoff vector $(17/10, 17/10)$. thus, the value of the game is 17/10.