Stat 155 Game theory, Yuval Peres **Fall 2004**

Lecture 3

Topic: Combinatorial games

Example We begin with K chips in one pile. Players I and II make their moves alternately, with player I going first. Each players takes between one and four chips on his turn. The player who removes the last chip wins the game. We write

$$\mathbf{N} = \{n \in \mathbb{N} : \text{ player } I \text{ wins if there } n \text{ chips at the start}\},$$

where we are assuming that each player plays optimally. We set $\mathbf{P} = \mathbb{N} - \mathbf{N}$. Clearly, $\{1,2,3,4\} \subseteq \mathbf{N}$, because player I can win with his first move. That $5 \in \mathbf{P}$ because the number of chips after the first move must lie in the set $\{1,2,3,4\}$. That $\{6,7,8,9\} \in \mathbf{N}$ follows from the fact that player I can force his opponent into a losing position by ensuring that there are five chips at the end of his first turn. Continuing this line of argument, we find that $\mathbf{P} = \{n \in \mathbb{N} : n \, \text{is divisible by five}\}$.

Definition 1 *A combinatorial game has two players, and a set, which is usually finite, of possible positions. There are rules for each of the players that specify the available legal moves for the player whose turn it is. If the moves are the same for each of the players, the game is called impartial. Otherwise, it is called partisan. The players alternate moves. Under normal play, the player who cannot move loses. Under mis'ere' play, the player who makes the final move loses.*

Definition 2 *Generalising the earlier example, we write* $\mathbf{N}$ *for the collection of starting positions from which the first player to move will win provided that each of the players adopts an optimal straegy, and* $\mathbf{P}$ *for the remaining starting positions.*

Example: nim. In this game, there are several piles, each containing finitely many chips. A legal move is to remove any positive number of chips from a given pile. The game has normal play, so the player who takes the final chip wins. We will write the state of play in the game in the form $(n_1, n_2 \ldots, n_k)$, meaning that there are k piles of chips still in the game, and that the first has n_1 chips in it, the second n_2, and so on.

Note that $(1,1) \in \mathbf{P}$, because the game must end after the second turn from this beginning. We see that $(1,2) \in \mathbf{N}$, because the first player can bring $(1,2)$ to $(1,1) \in \mathbf{P}$. Similarly, $(n,n) \in \mathbf{P}$ for $n \in \mathbb{N}$ and $(n,m) \in \mathbf{N}$ if $n,m \in \mathbb{N}$ are not equal. We see that $(1,2,3) \in \mathbf{P}$, because, whichever move the first player makes, the second can force there to be two piles of equal size. The following theorem characterises the sets N and P for this game:

Theorem 1 *Given a starting position $(n_1, n_2, \ldots, n_k)$, write each of the n_i in binary, and sum each of the columns mod 2. The position is in $\mathbf{P}$ if and only if all of the answers are zero.*

To illustrate the theorem, consider the starting position $(1,2,3)$. We find that

number of chips (decimal)	number of chips (binary)
1	01
2	10
3	11

Summing modulo two, the zeros and ones in the two columns of binary, we obtain 00. The theorem confirms that $(1,2,3) \in \mathbf{P}$.

Homework:

(i) the weatherman problem.

(ii) Consider a game where there are two piles of chips. Players may withdraw chips from exactly one of the piles on their turns, with the legal moves being to remove between one and four chips from the first pile, and from between one and five chips from the second pile. Determine for which $n,m \in \mathbb{N}$ it is the case that $(n,m) \in \mathbf{P}$.

(iii) *Nimble:* in this game, a finite number of coins are placed on a row of slots of finite length. Several coins can occupy any given slot. In any given turn, a player may move one of the coins to the left, by any number of places. The game ends when all the coins are at the leftmost slot.