Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 4,5,6

In the last lecture, we defined N and P positions for a combinatorial game. We will now show more formally that each starting position in a combinatorial game lies in either $\mathbf{N}$ or $\mathbf{P}$. Recall that the end position 0 of any game lies in $\mathbf{P}$ if the game is conducted under normal play. Any position for which there is a move to a P-position lies in $\mathbf{N}$. Any position where all moves lead to N positions lies in $\mathbf{P}$. Writing this more formally, we define

$$\begin{aligned} \mathbf{P}_0 &= \{0\} \\ \mathbf{N}_{i+1} &= \{\text{positions } x \text{ for which there is a move leading to } \mathbf{P}_i\} \\ \mathbf{P}_i &= \{\text{positions } y \text{ such that each move leads to } \mathbf{N}_i\}, \end{aligned} \tag{1}$$

for each $i \in \mathbb{N}$. We set

$$\mathbf{N} = \bigcup_{i \geq 0} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i \geq 0} \mathbf{P}_i.$$

Consider a game where, for each starting position x, the game must end within $B(x) < \infty$ moves, no matter which moves the two players make. We check by induction on $B(x)$ that all positions lies in $\mathbf{N} \cup \mathbf{P}$. If $B(x) = 0$, this is true, because $\mathbf{P}_0 \subseteq \mathbf{P}$. Assume the inductive hypothesis for those positions x for which $B(x) \leq n$, and consider any position z satisfying $B(z) = n + 1$. There are two cases to handle: the first is that each move from z leads to a position in $\mathbf{N}$ (that is, to a member of one of the previously constructed sets $\mathbf{N}_i$). Then z lies in one of the sets $\mathbf{P}_i$ and thus in $\mathbf{P}$. In the second case, there is a move from z to some P-position. This implies that $z \in \mathbf{N}$. Thus, all initial positions lie in either $\mathbf{N}$ or $\mathbf{P}$.

The metereologist question: recall this problem, from Lecture 2. By checking the derivative of $g_p(x)$, we see that $f(x) = \log x$ or $f(x) = -(1-x)^2$ are such that $g_p(p) > g_p(x)$ for $x \in [0,1] - \{p\}$.

Last time, we stated a theorem on how to play optimally the game of nim. We now present the proof.

Proof of Bouton's Theorem We write $n \oplus m$ to be the nim-sum of $n, m \in \mathbb{N}$. This operation is the one described in the statement of the theorem. We write n and m in binary, and compute the value of the sum of the digits in each column modulo 2. The result is the binary expression for the nim-sum

$n \oplus m$. Another way of saying this is that the nim-sum of a collection of values $(m_1, m_2, \ldots, m_k)$ is the sum of all the powers of two that occurred an odd number of times when we wrote each of the numbers m_i as a sum of powers of two. Here is an example: $m_1 = 13, m_2 = 9, m_3 = 3$. In powers of two:

$$\begin{array}{rcl} m_1 & = & 2^3 + 2^2 \qquad + 2^0 \\ m_2 & = & 2^3 \qquad\qquad + 2^0 \\ m_3 & = & \qquad\quad +2^1 + 2^0. \end{array}$$

In this case, the powers of two that appear an odd number of times are $2^0 = 1$ and $2^1 = 2$. this means that the nim-sum $m_1 \oplus m_2 \oplus m_3 = 1 + 2 = 3$. For the case where $(m_1, m_2, m_3) = (5, 6, 15)$, we write, in purely binary notation,

5	0	1	0	1
6	0	1	1	0
15	1	1	1	1
	1	1	0	0

making the nim-sum 12 in this case. We define $\hat{P}$ to be those positions with nim-sum zero, and $\hat{N}$ to be all other positions. We claim that

$$\hat{P} = \mathbf{P}, \text{ and } \hat{N} = \mathbf{N}.$$

To check this claim, we need to show two things. Firstly, that $0 \in \hat{P}$, and that, for all $x \in \hat{N}$, there exists a move from x leading to $\hat{P}$. Secondly, that for every $y \in \hat{P}$, all moves from y lead to $\hat{N}$.

Note firstly that $0 \in \hat{P}$ is clear. Secondly, suppose that $x = (m_1, m_2, \ldots, m_k) \in \hat{N}$. Set $s = m_1 \oplus \ldots \oplus m_k$. Writing each m_i in binary, note that there are an odd number of values of $i \in \{1, \ldots, k\}$ for which the binary expression for m_i has a 1 in the position of the left-most one in the expression for s. Choose one such i. Note that $m_i \oplus s < m_i$, because $m_i \oplus s$ has no 1 in this left-most position, and so is less than any number whose binary expression does have a 1 there. So we can play the move that removes from the i-th pile $m_i - m_i \oplus s$ chips, so that m_i becomes $m_i \oplus s$. The nim-sum of the resulting position $(m_1, \ldots, m_{i-1}, m_i \oplus s, m_{i+1}, \ldots, m_k)$ is zero, so this new position lies in $\hat{P}$. We have checked the first of the two conditions which we require.

To verify the second condition, we have to show that if $y = (y_1, \ldots, y_k) \in \hat{P}$, then any move from y leads to a position $z \in \hat{N}$. We write the y_i in binary:

$$\begin{array}{rcl} y_1 & = & y_1^{(n)} y_1^{(n-1)} \ldots y_1^{(0)} = \sum_{j=0}^{m} y_1^{(j)} 2^j \\ & & \ldots \\ & & \ldots \\ & & \ldots \\ y_k & = & y_k^{(n)} y_k^{(n-1)} \ldots y_k^{(0)} = \sum_{j=0}^{m} y_k^{(j)} 2^j. \end{array}$$

A particular example:

$$\begin{aligned} 4 &= 0100 = 2^2 \\ 6 &= 0110 = 2^2 + 2^1 \\ 15 &= 1111 = 2^3 + 2^2 + 2^1 + 2^0 \\ 13 &= 1101 = 2^3 + 2^2 + 2^0 \,. \end{aligned}$$

By assumption, $y \in \hat{P}$. This means that the nim-sum $y_1^{(j)} \oplus \ldots \oplus y_k^{(j)} = 0$ for each j. In other means, that $\sum_{l=1}^{k} y_l^{(j)}$ is even for each j. Suppose that we remove chips from pile l. We get a new position $z = (z_1, \ldots, z_k)$ with $z_i = y_i$ for $i \in \{1, \ldots, k\}$, $i \neq l$, and with $z_l < y_l$. (The case where $z_l = 0$ is permitted.) Consider the binary expressions for y_l and z_l:

$$\begin{aligned} y_l &= y_l^{(n)} y_l^{(n-1)} \ldots y_l^{(0)} \\ z_1 &= z_l^{(n)} z_l^{(n-1)} \ldots z_l^{(0)} \,. \end{aligned}$$

We scan these two rows of zeros and ones until we locate the first instance of a disagreement between them. In the column where it occurs, the nim-sum of y_l and z_l is one. This means that the nim-sum of $z = (z_1, \ldots, z_k)$ is also equal to one in this column. Thus, $z \in \hat{N}$. We have checked the second condition that we needed, and so, the proof of the theorem is complete. □

Example: the game of *rims*. In this game, a starting position consists of a finite number of dots in the plane, and a finite number of continuous loops. Each loop must not intersect itself, nor any of the other loops. Each loop must pass through at least one of the dots. It may pass through any number of them. A legal move for either of the two players consists of drawing a new loop, so that the new picture would be a legal starting position. The players' aim is to draw the last legal loop.

We can see that the game is identical to a variant of nim. For any given position, think of the dots that have no loop going through them as being divided into different classes. Each class consists of the set of dots that can be reached by a continuous path from a particular dot, without crossing any loop. We may think of each class of dots as being a pile of chips, like in nim. What then are the legal moves, expressed in these terms? Drawing a legal loop means removing at least one chip from a given pile, and then splitting the remaining chips in the pile into two separate piles. We can in fact split in any way we like, or leave the remaining chips in a single pile.

This means that the game of rims has some extra legal moves to those of him. However, it turns out that these extra make no difference, and so that the sets **N** or **P** coincide for the two games. We now prove this.

Thinking of a position in rims as a finite number of piles, we write P_{nim} and N_{nim} for the **P** and **N** positions for the game of nim (so that these sets were found in Bouton's Theorem). We want to show that

$$\mathbf{P} = P_{nim}, \text{ and that } \mathbf{N} = N_{nim} \tag{2}$$

where here, **P** and **N** refer to the game of rims.

What must we check? Firstly, that $0 \in \mathbf{P}$, which is immediate. Secondly, that from any position in N_{nim}, we may move to P_{nim} by a move in rims. This is fine, because each nim move is legal in rims. Thirdly, that for any $y \in P_{nim}$, any rims move takes us to a position in N_{nim}. If the move does not involve breaking a pile, then it is a nim move, so this case is fine. We need then to consider a move where y_l is broken into two parts u and v whose sum satisfies $u + v < y$. Note that the nim-sum $u \oplus v$ of u and v is at most than the ordinary sum $u + v$: this is because the nim-sum consists of the sum of the odd powers that appear in the expression for $u + v$ as a sum of powers of two, that is, it involves omitting from this expression certain powers of two. Thus,

$$u \oplus v \leq u + v < y_l.$$

So the rims move in question amounted to replacing the pile of size y_l by one with a smaller number of chips, $u \oplus v$. Thus, the rims move has the same effect as a legal move in nim, so that, when it is applied to $y \in P_{nim}$, it produces a position in N_{nim}. This is what we had to check, so we have finished proving (2).