

Lectures 9,10

The solution of staircase nim. Recall staircase nim, question 6 of page I – 13 of Ferguson. We claim that a configuration is a P -position in staircase nim if the numbers of coins on odd-numbered steps forms a P -position in nim. To see this, note that moving coins from an odd-numbered step to an even-numbered one represents a legal move in a game of nim consisting of piles of chips lying on the odd-numbered steps. We need only check that moving chips from even to odd numbered steps is not useful. A player who has just seen his opponent to do this may move the chips newly arrived at an odd-numbered location to the next even-numbered one, that is, he may repeat his opponent's move at one step lower. This restores the nim-sum on the odd-numbered steps to its value before the opponent's last move. This means that the extra moves can play no role in changing the outcome of the game from that of nim on the odd-numbered steps.

Moore's nim_k : in this game, recall that players are allowed to remove any number of chips from at most k piles in any given turn. We write the binary expansions of the pile sizes (n_1, \dots, n_l) :

$$\begin{aligned} n_1 &= n_1^{(m)} \dots n_1^{(0)} = \sum_{j=0}^m n_1^{(j)} 2^j \\ &\cdot \\ &\cdot \\ &\cdot \\ n_l &= n_l^{(m)} \dots n_l^{(0)} = \sum_{j=0}^m n_l^{(j)} 2^j \end{aligned}$$

We set

$$\hat{P} = \left\{ (n_1, \dots, n_l) : \sum_{i=1}^l n_i^{(r)} = 0 \pmod{k+1} \text{ for each } r \geq 0. \right\}$$

Theorem 1 (Moore's theorem) *We have that $\hat{P} = \mathbf{P}$.*

Proof: Firstly, note that the terminal position 0 lies in \hat{P} . There are two other things to check: firstly, that from \hat{P} , any legal move takes us out of there. To see this, take any move from a position in \hat{P} , and consider the leftmost column l for which we change the binary expansion of one of the

pile numbers. Any change in this column must be from one to zero. The existing sum of the ones and zeros mod $k+1$ is zero, and we are adjusting at most k piles. Since ones are turning into zeros, and at least one of them is changing, we could only get back to $0 \pmod{k+1}$ in this column were we to change $k+1$ piles. This isn't allowed, so we have verified that no move from \hat{P} takes us back there.

We must also check that for each position in \hat{N} (which we define to be the complement of \hat{P}), there exists a move into \hat{P} . This step of the proof is a bit harder. How to select the k piles from which to remove chips? Well, we work by finding the leftmost column whose mod $k+1$ sum is not-zero. We select any r rows with a one in this column, where r is the number of ones in the column reduced mod $k+1$ (so that $r \in \{0, \dots, k\}$). We've got the choice to select $k-r$ more rows if we need to. We do this moving to the next column to the right, and computing the number s of ones in that column, ignoring any ones in the rows that we selected before, and reduced mod $k+1$. If $r+s < k$, then we add s rows to the list of those selected, choosing these so that there is a one in the column currently under consideration, and different from the rows previously selected. If $r+s \geq k$, we choose $k-r$ such rows, so that we have a complete set of k chosen rows. In the first case, we still need more rows, and we collect them successively by examining each successive column to the right in turn, using the same rule as the one we just explained. Let l denote the number of the column whose examination led us to include rows in such a way that we had the total of k , where we count the columns from the right. The point of doing this is that we have chosen the rows in such a way that, for any column, either that column has no ones from the unselected rows because in each of these rows, the most significant digit occurs in a place to the right of this column, or the mod $(k+1)$ sum in the rows other than the selected ones is not zero. If a column is of the first type, we set all the bits to zero in the selected rows. This gives us complete freedom to choose the bits in the less significant places. In the other columns, we may have say $t \in \{1, \dots, k\}$ as the mod $(k+1)$ sum of the other rows, so we choose the number of ones in the selected rows for this column to be equal to $k-t$. This gives us a mod $(k+1)$ sum zero in each row, and thus a position in \hat{P} . This argument is not all that straightforward, it may help to try it out on some particular examples: choose a small value of k , make up some pile sizes that lie in \hat{N} , and use it to find a specific move to a position in \hat{P} . Anyway, that's what had to be checked, and the proof is finished. \square